PROPOSAL FOR MANIPULATION & OPERATION RULES ON INFINITE SERIES

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San Diego, 29 December 2021

"Only in the eyes of love you can find infinity." – Sorin Cerin

ABSTRACT:

When dealing with infinite series, one common question comes up: "Do the known arithmetic rules work for infinite series?" For example, do commutative and associative properties still work?

This is a statement which has been widely acknowledged: If a series converges absolutely, then no matter how you rearrange the terms, the sum will be the same. We can manipulate it with the ordinary arithmetic, but if a series is conditionally convergent series, it can be rearranged so that it converges to a different number, in fact, to any different number, or even diverge to ±∞ (Riemann Rearrangement Theorem).

For example: We know that the alternating harmonic series conditionally converges. Its sum is the natural log of 2.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = ln2$$

But according to Riemann, the series can be rearranged so that it can converge to $\frac{1}{2}ln2$ or $\frac{3}{2}ln2$ (see example 1 in Rule 2), etc. This leads to weird statement like this:

½ = 1 = 1.5 = 2 = …

It gets even weirder if we split the alternating harmonic series into positive part and negative part:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots = \infty \qquad \text{(a)}$$

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{19} - \cdots = -\infty \qquad \text{(b)}$$

and by using a special arrangement of the alternating harmonic series as follows:

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{149} + \frac{1}{151}\right) + \left(-\frac{1}{2} + \frac{1}{153} + \cdots + \frac{1}{409}\right) + \left(-\frac{1}{4} + \frac{1}{411} + \cdots\right) + \cdots = \pi \quad \text{(c)}$$

Value of the expression in the first parenthesis is 3.1471. Adding the second parenthesis to it, the result will be 3.1432 (closer to pi). Adding the third parenthesis, the result will be 3.1415 (more closer to pi), and so on. From a, b, and c, we can have a statement like this:

$$\infty - \infty = \boldsymbol{\pi}$$

To avoid uncomfortable situations like these, I propose three rules for manipulating and operating on infinite series.

RULE ONE: *Every infinite series has **a pseudo length** {L} (or **a pseudo cardinality** {L} for every infinite set). The value of this pseudo length is arbitrary, but it is a basic property of the series (or the set) and it's useful for checking wrong manipulations or operations on series (or sets).*

Caution: Georg Cantor's countable, uncountable infinite concepts as well as Ordinality terminology can no longer be used because they violate this rule (see corollary below). The Continuum Hypothesis will be disproved under this new rule of mathematics.

Examples of pseudo length and pseudo cardinality:

$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + \cdots + \infty$ If we assign a pseudo length **{L}** for this series

Then $1 + \sum_{n=1}^{\infty} n$ will have a pseudo length of **{L+1}**

If {**L**} is the pseudo cardinality of the natural numbers set, then the pseudo cardinality of the whole number set is {**L+1**}; the pseudo cardinality of all even integers set, or all odd integers set is {**L/2**}; the pseudo cardinality of the integer number set is {**2L + 1**}.

Under this rule, we have:

{**N**} < {**W**} < {**Z**} < {**Q**} < {**R**} < {**C**}

RULE TWO: *Two series look similar but have different pseudo lengths are not the same.*

Example 1: (Riemann Rearrangement Theorem, changing sum example)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \boldsymbol{ln2}$$

Riemann rearranged:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \cdots = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \cdots$$

The series still have pseudo length {L}, but the following resulting series has pseudo length {**L/2**}:

$$= (\frac{1}{2}) - \frac{1}{4} + (\frac{1}{6}) - \frac{1}{8} + (\frac{1}{10}) - \cdots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right) = \frac{1}{2}ln2$$

Consider the series inside of the above final parenthesis as the original series (ln2) is wrong.

<u>We can figure out the result like this</u>: The original series lost half of its terms along the way to infinity, so its sum must be (1/2)ln2.

This claim is also wrong: $\mathbf{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \frac{3}{2}\boldsymbol{ln2}}$

Because of this manipulation:

$$\left\{\begin{array}{ll} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots = ln2 & \{L\} \\ \frac{1}{2} \qquad - \frac{1}{4} \qquad + \frac{1}{6} \qquad - \frac{1}{8} + \cdots = \frac{1}{2}ln2 & \{L\} \end{array}\right\}$$

The two statements above are correct, but line them up as above and add them together to get the following statement is wrong (violate rule 3: shifting).

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \frac{1}{9} - \cdots = \frac{3}{2}ln2 \quad \{L + \frac{L}{2} = \frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{L}\}$$

<u>We can figure out the result like this</u>: the original series with pseudo length {L} and sum ln2, was added to half of itself with pseudo length {L/2} and sum (1/2)ln2. Although the resulting series looks alike the original series, but actually it’s different, its pseudo length is (3/2)L and of course its sum is (3/2)ln2.

It’s incorrect to say that “*Conditionally convergent series can be rearranged so that it converges to a different number, in fact, to any different number, or even diverge to* ±∞”. When we rearrange the series, we modify the original series. We get different modified series because of these rearrangement techniques. These different modified series look alike the original series and can converge to different values (or diverges), but they are **NOT** the original series.

Example 2: (Cesàro summation assigns <u>Grandi's divergent series</u>)

$$\sum_{n=1}^{\infty}(-1)^{n-1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots ..$$

This claim is wrong:

$$\sum_{n=1}^{\infty}(-1)^{n-1} = \frac{1}{2}$$

Because of this manipulation:

Let S = 1 – 1 + 1 – 1 + 1 – 1 + ….   Pseudo length {L}

Now subtract S from 1:

1 – S = 1 – (1 – 1 + 1 – 1 + 1 – 1 + …)   This series has pseudo length {L+1}
= 1 – 1 + 1 – 1 + 1 – 1 + .. = S → 2S = 1 → S = ½

This claim is wrong because the two series S and (1-S) have different pseudo lengths.

Example 3: This claim is wrong: **1 + 2 + 4 + 8 + 16 + … = - 1**

Because of this manipulation

Let S = 1 + 2 + 4 + 8 + 16 + … {L}
= 1 + 2(1 + 2 + 4 + 8 + …)
the series inside parenthesis has a pseudo length of {L-1}
= 1 + 2S → S = -1

Example 4: The following claim is on the same summation as Ramanujan summation, but it has different result (-1/8) instead of (-1/12).

This claim is wrong: **1 + 2 + 3 + 4 + 5 + … = - 1/8**

Because of this manipulation

Let S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + … {L}
= 1 + (2 + 3 + 4) + (5 + 6 + 7) + (8 + 9 + 10) + … {L}
= 1 + 9 + 18 + 27 + … {(L-1)/3 + 1}
= 1 + 9(1 + 2 + 3 + …)
= 1 + 9S => S = -1/8

The series inside parenthesis looks alike with the original series, but it has different pseudo length {(L-1)/3}. We can't assign it as S.

If we can accept Ramanujan summation's manipulation (see example 2 in Rule 3), then what's wrong with this manipulation?

Corollary: Set Z and set Q do not have the same pseudo cardinality as N.

Let {**L**} be the pseudo cardinality of the natural numbers set N = {1, 2, 3, 4, 5, …}. Then the set of all even integer numbers
N_{even} = {2, 4, 6, 8, …} will have the pseudo cardinality of {**L/2**}
If we consider the "double integer number" set 2N = {2, 4, 6, 8, 10, …}, its pseudo cardinality is {**L**} (same as N). This set is different with N_{even} set (although they look the same) because they have different pseudo cardinalities. There is no ***Bijection*** between 2N set and N_{even} set.
We can use similar arguments to show that Z and Q do not have the same cardinality as N.
Ordinality terminology ω can no longer be used because (**no bijection**):
If ordinality of {0, 1, 2, 3, 4, …} = ω
Ordinality of {1, 2, 3, 4, … 0} = ω not (ω +1)
Ordinality of {2, 3, 4, … 0, 1} = ω not (ω+2)
Ordinality of {0, 2, 4, … 1, 3, 5, …} = ω not (2 ω)

RULE THREE: *Add or subtract two infinite series if and only if they have the same pseudo lengths and positions of terms are in correct order (**no shifting terms**). The resulting series must sustain the same pseudo length*.

Correct operation example:

$\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + 10 + \cdots + \infty$ {L}
$\sum_{n=1}^{\infty} n \quad = 1 + 2 + 3 + 4 + 5 + \cdots + \infty$ {L}

Add the two expressions, we get:

$\sum_{n=1}^{\infty} 2n + \sum_{n=1}^{\infty} n = 3 + 6 + 9 + 12 + 15 + \cdots + \infty = \sum_{n=1}^{\infty} 3n$ {L} **Correct**!

Subtract the two expressions, we get:

$\sum_{n=1}^{\infty} 2n - \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + \cdots + \infty = \sum_{n=1}^{\infty} n$ {L} **Correct**!

Wrong operations:

Example 1:
$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + \cdots + \infty$ {L}
$\sum_{n=1}^{\infty} 2n = \quad 2 \quad + 4 \quad + 6 \ldots + \infty$ {L} **(shifting)**

Subtract the two expressions to get:

$\sum_{n=1}^{\infty} n - \sum_{n=1}^{\infty} 2n = 1 + 3 + 5 + \cdots + \infty = \sum_{n=1}^{\infty}(2n - 1)$ **violate rule 3**!

Example 2: (Ramanujan summation)

This claim is wrong: **1 + 2 + 3 + 4 + 5 + … = - 1/12**

Because of this manipulation:

Let S = 1 + 2 + 3 + 4 + 5 + 6 + …
4S = 4 + 8 + 12 + … **(shifting)**

Subtract the two expressions to get:
-3S = 1 – 2 + 3 - 4 + 5 – 6 + …
= ¼ hence: S = -1/12 **violate rule 3**!

Note: This statement had been used in String Theory (Hardy-Ramanujan equation for Partitions and the Hagedorn Temperature). From Example 4 in Rule 2, one can argue that: "***Why don't we use 1 + 2 + 3 + 4 + … = -1/8 ?***"
Assume that there is nothing wrong with Ramanujan summation, then we can manipulate:

2 + 4 + 6 + 8 + 10 + … = 2(1 + 2 + 3 + 4 + 5 + …) = 2(-1/12) = -2/12

Then from: 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + … = -1/12
and 2 + 4 + 6 + 8 + … = -2/12

I can get: 1 + 3 + 5 + 7 + 9 + … = -1/12 – (- 2/12) = 1/12

Can this statement:

1 + 3 + 5 + 7 + 9 + … = 1/12

become popular as **1 + 2 + 3 + 4 + 5 + … = - 1/12 ?**

It leads to another weird statement:

2 + 4 + 6 + 8 + ... = -1/6

1 + 3 + 5 + 7 + ... = 1/12

As a result: **-1/6 > 1/12** **(negative # > positive #)**

If we subtract the above two statements as follows

$$-\begin{cases} 2+4+6+8+\ldots = -1/6 \\ 1+3+5+7+\ldots = 1/12 \end{cases}$$

1 + 1 + 1 + 1 + ... = - 1/4

The series on the left is divergent: **1 + 1 + 1 + 1 + … = +∞** since its sequence of partial sums increases monotonically without bound.

If we use Riemann zeta function regularization as the value of s = 0 and apply the analytic continuation, then we should get -1/2 not -1/4:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s),$$

$$\Gamma(n) = (n-1)! \quad \Gamma(1) = 1$$

$$\zeta(0) = \frac{1}{\pi} \lim_{s\to 0} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) = \frac{1}{\pi} \lim_{s\to 0} \left(\frac{\pi s}{2} - \frac{\pi^3 s^3}{48} + \ldots\right) \left(-\frac{1}{s} + \ldots\right) = -\frac{1}{2}$$

$$1 + 1 + 1 + 1 + \cdots = \zeta(0) = -\frac{1}{2}.$$

Can we accept this statement?

-1/2 = -1/4

How about this arrangement?

1 + 3 + 5 + 7 + 9 + 11 + … = 1 + (3 + 5) + (7 + 9) + (11 + 13) + …
= 1 + 8 + 16 + 24 + …
= 1 + 8(1 + 2 + 3 + 4 + …)
= 1 + 8(-1/12) = 1 – 2/3 = **1/3**

How about this arrangement?

S = 1 + 3 + 5 + 7 + 9 + 11 + 13 + … = (1 + 3) + (5 + 7) + (9 + 11) + …
= 4 + 12 + 20 + …
= 4(1 + 3 + 5 + …) = 4S

hence S = **1 + 3 + 5 + 7 + 9 + 11 + 13 +… = 0**

Note: This series (involved in free-fall motion, physics) is not a conditionally convergent series, but we still can rearrange it and make it "converges" to different values (without using three proposed rules).

Example 3: (Using Zeta function and Eta function to derive **1+2+3+4+… = -1/12**)

Let $\xi(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$ {**L**}

And $\eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$ {L}

Subtract the two functions to get

$$\xi(s) - \eta(s) = 0 + \frac{2}{2^s} + 0 + \frac{2}{4^s} + 0 + \frac{2}{6^s} + 0 + \cdots \quad \{L\}$$

$$= \frac{2}{2^s}\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) \quad \{L/2\}$$

$$= \frac{2}{2^s}\xi(s) \quad \rightarrow \quad \xi(s) = \frac{\eta(s)}{[1-\frac{2}{2^s}]}$$

let s = -1 we get: **1+2+3+4+… = -1/12**

We can't assign Zeta function for the series inside the above parenthesis because it has different pseudo length (violate rule 2).

Example 4: (Using **Zeta function** to derive **Euler Product**)

$$\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \quad \{L\}$$

Multiply $\xi(s)$ by $-\frac{1}{2^s}$ to get:

$$-\frac{1}{2^s}\xi(s) = -\frac{1}{2^s} - \frac{1}{4^s} - \frac{1}{6^s} - \frac{1}{8^s} - \cdots \quad \{L\}$$

Add the above two expressions with the line up as follows (purpose: remove all elements that have a factor of prime number 2 on the right side):

$$\xi(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

$$-\frac{1}{2^s}\xi(s) = \quad -\frac{1}{2^s} \quad\quad -\frac{1}{4^s} \quad\quad -\frac{1}{6^s} \quad\quad -\frac{1}{8^s} \quad\quad - \cdots$$ **shifting!**

and get:

$\xi(s)\left(1-\frac{1}{2^s}\right)=\frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\cdots$ **violate rule 3**!

Therefore, we can't continue to apply this process to remove all elements that have a factor of prime number 3, prime number 5, prime number 7, etc. on the right side anymore.

As a result, the above method to derive the following formula is unacceptable:

$$\xi(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\cdots=\prod_{p_i}^{\infty}\frac{1}{(1-\frac{1}{p_i{}^s})}$$

CONCLUSION:

By using these three simple rules for manipulating and operating on infinite series, we can avoid accepting many unpleasant mathematical statements, especially for conditionally convergent series.

However, who dares to question the legitimacy of these famous formulae Euler product for Riemann Zeta function, Riemann Rearrangement Theorem, Cesàro summation, Ramanujan summation or the famous theories which adapted those formulae (example, example)?

I hope some readers can see my point in this article: We need some new regularizations or new theory for this "Wild West area of mathematics" – infinite series.

IS A CONVERGENT SERIES FINITE?

ABSTRACT:

We know that a finite series is convergent. But how about this statement?
"If a series is convergent, it is a finite series."

DEBATE:

Let's consider this series.

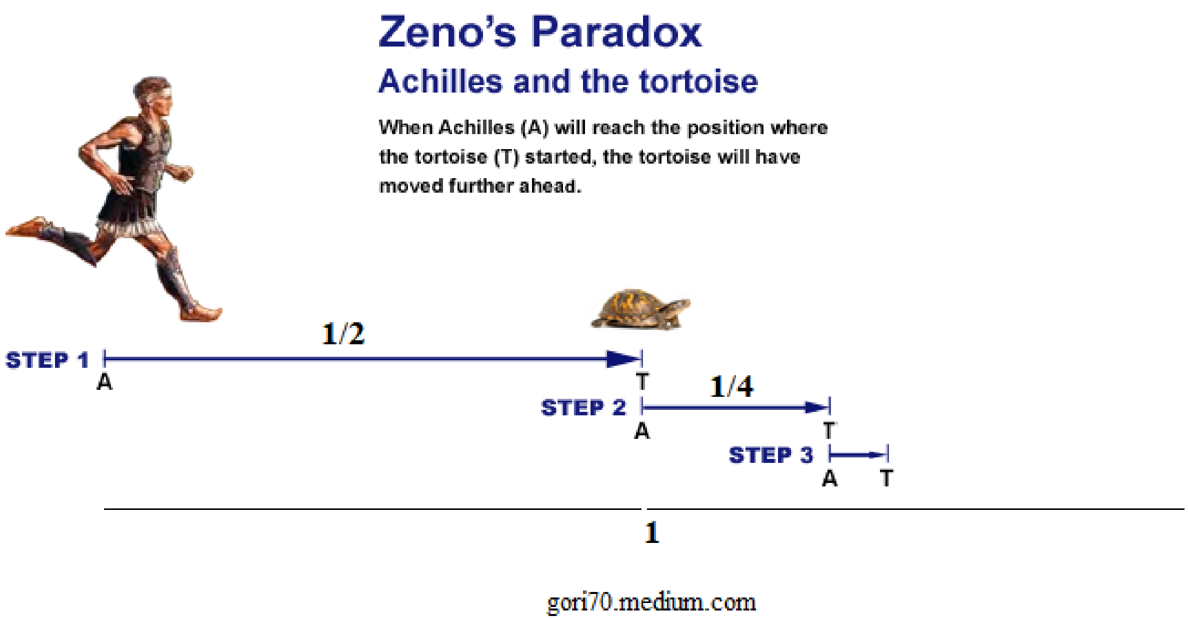
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots=\frac{1}{2}+\frac{1}{2}.\frac{1}{2}+\frac{1}{2}.\frac{1}{4}+\frac{1}{2}.\frac{1}{8}+\frac{1}{2}.\frac{1}{16}+\cdots=1$$

It's a convergent series.

Except the first term, every term in the series is equal half of the previous term. The series is related to the Greek philosopher Zeno's paradoxes (nearly 2,500 years ago).

Achilles, the fleet-footed hero of the Trojan War, is engaged in a race with a lowly tortoise, which has been granted a head start. Achilles' task initially seems easy, but he has a problem. Before he can overtake the tortoise, he must first catch up with it. While Achilles is covering the gap between himself and the tortoise that existed at the start of the race, however, the tortoise creates a new gap. The new gap is smaller than the first, but it is still a finite distance that Achilles must cover to catch up with the animal. Achilles then races across the new gap.

To Achilles' frustration, while he was scampering across the second gap, the tortoise was establishing a third. The upshot is that Achilles can never overtake the tortoise. No matter how quickly Achilles closes each gap, the slow-but-steady tortoise will always open new, smaller ones and remain just ahead of the Greek hero (https://slate.com/technology/2014/03/zenos-paradox-how-to-explain-the-solution-to-achilles-and-the-tortoise-to-a-child.html).



We can prove the series converges to 1 by using several different mathematical techniques such as:

- Geometric series formula:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{a_1}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$$

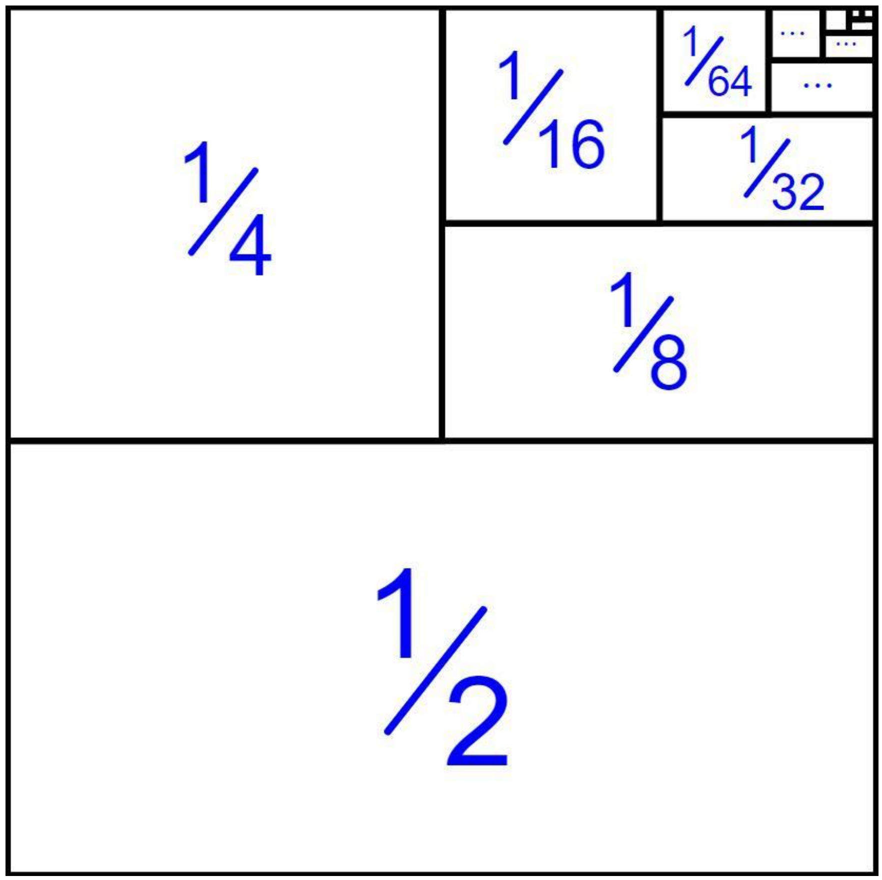
- Algebra manipulation trick:

$$S=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

$$2S=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

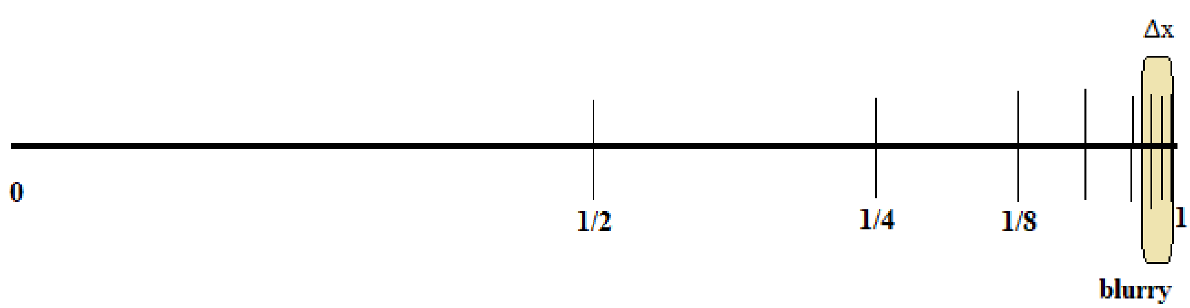
$$2S-S=1 \quad \text{hence} \quad S=1$$

- Geometry trick:

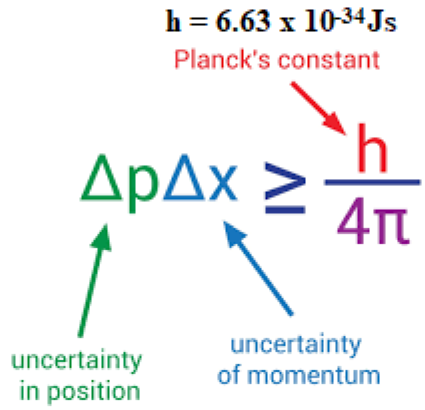


or we can use Quantum Zeno effect in physics to show the series is convergent.

Although above methods can prove the convergence of the series, but they can't answer the question: how many steps or how long for the Achilles to catch the tortoise?
However, in this article we don't focus on that matter. What we want to know is that the series is finite or infinite.
Let's consider the following photo, vertical lines indicate the position of the Achilles during his run from the starting point 0 to 1 (target). As he approaches closer to the target, the uncertainty of finding him at the target becomes smaller and smaller.



According to the Heisenberg's uncertainty principle



If Δx is too small, Δp is larger. As a result, we can't see the vertical lines clear anymore, thing is getting blurrier. In another words, we can't continue to add tiny term ($1/2^n$) into the series forever. At some point, Heisenberg's uncertainty principle will be violated. That means there is a limitation of number of terms for the series. **The series is finite**.

What can we take away from here?

All convergent p – series are finite. $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots$ $(p > 1)$

For divergent infinite series with terms getting larger, they are still divergent infinite series.

$$1 + 2 + 3 + 4 + 5 + 6 + \ldots$$

But for "divergent infinite series" with terms getting smaller, we should **reconsider** them as convergent finite series. The harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$ (**To be continued**)