Cofactor Expansion and the Determinant's Role in 2D Area Transformation

1. Introduction to Determinants

A **determinant** is a scalar value that can be computed from the elements of a square matrix. It provides important properties of the matrix, such as invertibility, and plays a crucial role in solving linear systems, among other applications."

2. Cofactor Expansion Method

Before we proceed, let's recall two important definitions:

- **Minor** (M_{ij}): The determinant of the submatrix that remains after removing the i^{th} row and j^{th} column.
- **Cofactor** (C_{ij}): Defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

Cofactor Expansion Formula:

For an $n \times n$ matrix A, the determinant can be calculated by expanding along any row or column:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Example: Calculate the determinant of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & -2 \\ -1 & 0 & 3 \end{bmatrix}$$

Step 1: Choose a Row or Column

Typically, we choose the row or column with the most zeros to simplify calculations. Here, let's expand along the **first row**.

Step 2: Compute the Cofactors

- For $a_{11} = 2$:

  Remove the first row and first column:

$$M_{11} = \begin{vmatrix} 5 & -2 \\ 0 & 3 \end{vmatrix} = (5)(3) - (-2)(0) = 15$$

  Compute the cofactor:

$$C_{11} = (-1)^{1+1} M_{11} = (1)(15) = 15$$

- For $a_{12} = 3$:

  Remove the first row and second column:

$$M_{12} = \begin{vmatrix} 4 & -2 \\ -1 & 3 \end{vmatrix} = (4)(3) - (-2)(-1) = 12 - 2 = 10$$

  Compute the cofactor:

$$C_{12} = (-1)^{1+2} M_{12} = (-1)(10) = -10$$

- For $a_{13} = 1$:

  Remove the first row and third column:

$$M_{13} = \begin{vmatrix} 4 & 5 \\ -1 & 0 \end{vmatrix} = (4)(0) - (5)(-1) = 0 + 5 = 5$$

  Compute the cofactor:

$$C_{13} = (-1)^{1+3} M_{13} = (1)(5) = 5$$

Step 3: Compute the Determinant

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(A) = (2)(15) + (3)(-10) + (1)(5) = 30 - 30 + 5 = 5$$

3. Application: Determinant as Area Scaling Factor

Now, let's explore how this determinant relates to area scaling in 2D transformations.

Consider a linear transformation represented by a 2×2 matrix T, which transforms vectors in 2D space.

The absolute value of the determinant of T gives the factor by which areas are scaled under this transformation.

Example:

Let's take matrix:

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Step 1: Compute the Determinant

$$\det(T) = (2)(1) - (1)(1) = 2 - 1 = 1$$

Step 2: Interpret the Result

"The determinant is **1**, which means that the area of any shape after transformation remains the same —the transformation preserves area.

But let's see this visually with a specific shape.

Consider the unit square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

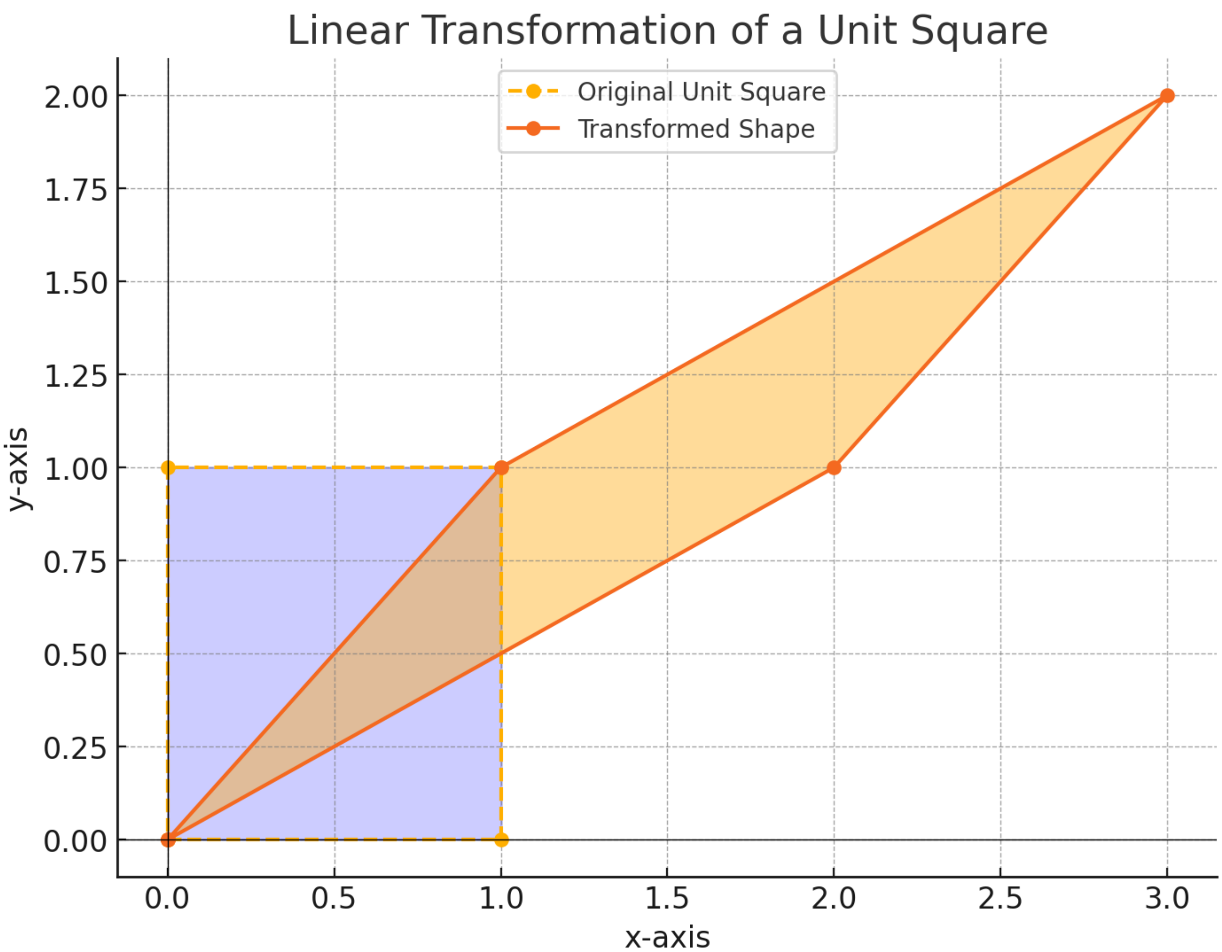
Step 3: Apply Transformation to Each Vertex

- $(0,0) \rightarrow T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $(1,0) \rightarrow T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $(1,1) \rightarrow T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $(0,1) \rightarrow T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Here is a visualization of the transformation:

- The **blue dashed square** represents the original unit square before the transformation.
- The **orange parallelogram** represents the transformed shape after applying the matrix T

This demonstrates how the linear transformation changes the shape while preserving the area since the determinant of T is 1.



4. Conclusion:

- We learned how to compute the determinant of a matrix using cofactor expansion.
- We saw that the determinant provides not just an algebraic value but also geometric insight.
- Specifically, in 2D, the determinant tells us how the area scales under a linear transformation.